S.

Let $\mathcal{B}(V)$ be a rank n Nichols algebra, diagonal type.

Let $E_0 := (e_1, \cdots, e_n)$ denote the standard basis of $\mathbb{Z}^n$

Define $W(V) :=$

$\{ (s, E) \mid s \in \mathrm{Aut}(\mathbb{Z}^n)$, $E$ is an ordered basis of $\mathbb{Z}^n$,

then $\exists\, m_1, m_2 \in \mathbb{N}_0$, $m_1 \leq m_2$,

and $i_1, \cdots, i_{m_2} \in \{1, 2, \cdots, n\}$

s.t. $\boxed{\underline{S_{i_m} \cdots S_{i_1}}(E_0)}$ is ~~well~~ well~~ed~~-defined for all $m \leq m_2$

$S_{i_{m_1}} \cdots S_{i_1}(E_0) = E$, $s = S_{i_{m_2}} \cdots S_{i_{m_1+1}}$

Cor: If $\mathcal{B}(V)$ is a ---- satisfy (F2)

then $W(V)$ is finite.

(F2) — $\Delta(\mathcal{B}(V))$ is finite.

a fact: the orbit of $E_0$ ~~under~~ is finite

$\forall\, S_{i_m} \cdots S_{i_1}$,

$S_{i_m} \cdots S_{i_1}(E_0) \subseteq \underline{\Delta(\mathcal{B}(V))}$

$\mathcal{B} \xrightarrow{S_i} \underline{\underline{\mathcal{B}_i \xrightarrow{S_j} \mathcal{B}_{i,j}}}$

$\underset{V}{s_i}(e_1, \cdots, e_n) = (e_1, \cdots, e_n)\boxed{(S_j)}$

$\in \Delta(\mathcal{B}_i)$ (prop1)

$S_i((e_1, \cdots, e_n)(S_j))$

$= (e_1, \cdots, e_n)(S_i)(S_j) \in \Delta(\mathcal{B}(V))$

Since $s_i(\Delta(B_i)) \subset \Delta(B(w))$

(prop1)

Cartan type

$V$ — diagonal type & $(q_{ij})$

satisfying $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$ $i,j \in \{1,2,\cdots,n\}$

where $a_{ii}=2$, $a_{ij}$ is a nonpositive integer $i\neq j$.

It's then always assumed that

$a_{ij}$ are maximal with the above properties.

Lemma 1.

Assume $V$ - Cartantype. $\underset{\text{Cartan matrix}}{(a_{ij})}$

(i) Suppose that $i\in\{1,2,\cdots,n\}$, $m\in\mathbb{N}_0$ s.t

$m<-a_{ij}$ for at least one $j\in\{1,2,\cdots,n\}$

then ~~$q$~~ $q_{ii}^{m+1}\neq 1$

(ii) For the numbers in $\underline{\text{Equation (12)}}$

$$m_{ij} := \min\{m\in\mathbb{N}_0 \mid \underline{(m+1)_{q_{ii}}}(q_{ii}^{m} q_{ij} q_{ji}-1)=0\}$$

one obtains $m_{ij}=-a_{ij}$ $\forall i,j\in\{1,2,\cdots,n\}$

Proof:

(i)

we have $q_{ii}^{a_{ij}} = q_{ij}q_{ji}$

Assume $\underline{m+1\le -a_{ij}}$ and $q_{ii}^{m+1}=1$

then $q_{ii}^{m+1+a_{ij}} = q_{ij}q_{ji}$, $\underline{\boxed{m+1+a_{ij}}\le 0}$

↯ to the choice $a_{ij}$

Thm 1:

Let $V$ be an $n$-dimensional YD-module of Cartan type with Cartan martix $C = (a_{ij})$

(i) If $C$ is not of finite type then $\delta(\mathcal{B}(V))$ is infinite.

(ii) If $C$ is of finite type, then $\delta(\mathcal{B}(V))$

Proof of (i):

Assume $\delta(\mathcal{B}(V))$ is finite.

By prop 1, $\forall i \in \{1, 2, \dots, n\}$ $\exists R_i \cong B(V_i)$

by lemma 1 (ii) one has $m_{ij} = -a_{ij}$

Choose a basis of $V_i$ s.t the $j$th basis vector $(\mathrm{ad}_c x_i)^{-a_{ij}}(x_j)$ $j \neq i$,

$y_i^R$ if $j = i$

Let $\{ \xi(i)_{jm} \mid j, m \in \{1, 2, \dots, n\} \}$

denote the set of structure constant of the braiding of $V_i$ respect to this basis.

By equation (17) one gets

$$q(i)_{jm} = \begin{cases} \cdots & m=j=i \\ & \underline{j=i,\ m\neq i} \\ & j\neq i,\ m=i \\ & \underline{j\neq i,\ m\neq j} \end{cases}$$

$$c\left(y_i^R \otimes (\mathrm{ad}_b x_i)^{-a_{im}}(x_m)\right)$$

$$= \underbrace{q_i^{-1}\cdot\left((\mathrm{ad}_b x_i)^{-a_{im}}(x_m)\right)}_{\shortparallel} \otimes y_i^R$$

$$q_{ii}^{a_{im}}\, q_{im}^{-1} = q_{mi}$$

$j\neq i,\ m\neq j$

In particular,

$$\begin{cases} q(i)_{jj} = q_{jj} \\ q(i)_{jm}\, q(i)_{mj} = q(i)_{jj}^{a_{jm}} \quad \forall j,m. \end{cases}$$

Here $B_i$ is of the same Cartan type

This means that the linear map $S$ in $(\xi E)$ as $B(V)$ don't depend on the basis $E$ $\in W(V)$ of $\mathbb{Z}^n$

By (F2) and cor1, $W(V)$ is finite.

If $C$ is symmetrizable then $W(V) \simeq W \times B$

$B$: orbit of $E_0$

$W$: Weyl group of $C$

备记:

Cartan type $W(V) \simeq G \times B$

Symmetrizable $\simeq W \times B$

It's well-known that $W$ is finite $\iff$ finite type.

Thus for (i) it suffices to show that the group $W(V)$ is infinite when $C$ is not symmetrizable

Note that if $W(V')$ corresponding to a $Y(1)$-module sub $V'$ of $V$ is infinite then $W(V)$ is infinite.

Further

if the Dynkin diagram associated to a Cartan matrix is simply-laced or has no cycles then it is symmetrizable

Kac P27, exercise 2.1

The matrix $A = (a_{ij})$ is symmetrizable $\iff$ $a_{ij} = 0 \Rightarrow a_{ji} = 0$ and

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} \textcircled{=}$$

$$a_{i_2 i_1} a_{i_3 i_2} \cdots a_{i_1 i_n}$$

Thus we only have to show that

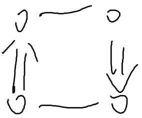
$W(V)$ is infinite if the corresponding dynkin diagramm is a cycle which is not simply-laced.

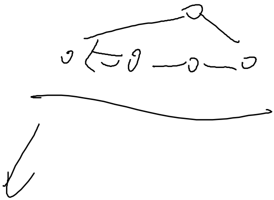
Further if we remove a node from the cycle, then we again to the symmetrizable case,

Thus it is sufficient to consider cycles (not symmetrizable) s.t after removing an arbitrary node the resulting diagram is of finite type.

① Such cycles has three nodes

② $\begin{pmatrix} 2 & -2 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ -1 & 0 & -1 & 2 \end{pmatrix}$ ③ $\begin{pmatrix} \text{- - -} \end{pmatrix}$





we have $\varepsilon_{11}^2 = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{44} = \varepsilon_{55}$
$= \varepsilon_{11}$

$\Rightarrow \underline{\varepsilon_{12}\varepsilon_{21} = \varepsilon_{11} = 1}$ $\varepsilon_{ii} \neq 1$

Consider ① :

the matrices $t_i$ of $s_i$, $i \in \{1,2,3\}$
with respect to $\{e_1, e_2, e_3\}$

are $t_1 = \begin{pmatrix} -1 & -a_{12} & -a_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $t_2 = \begin{pmatrix} 1 & 0 & 0 \\ -a_{21} & -1 & -a_{23} \\ 0 & 0 & 1 \end{pmatrix}$

$t_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a_{31} & -a_{32} & -1 \end{pmatrix}$

Without loss of generality, suppose $a_{12} < -1$

$s_1 \underline{s_2}\, \underline{s_3}$ has the matrix $\underline{t_1 t_2 t_3} = (\cdots)$

$$\boxed{s_3 s_2 s_1}(E_0) = s_3 s_2\left(\boxed{(e_1, \dots, e_n) t_1}\right)$$

$\in \mathrm{Aut}(\mathbb{Z}^n)$

$$= s_3\left(\underline{(e_1, \dots, e_n) t_1}\ \underline{t_2}\right)$$

$$= (e_1, \dots, e_n)\boxed{t_1 t_2 t_3}$$

$(\underline{s_3 s_2 s_1}, E_0)$

has finite order

Thus all eigenvalues of $t_1t_2t_3$ have to have absolute value 1,

hence trace $\leq 3 \Rightarrow 3 \rightarrow$ id

$$tr(t_1t_2t_3) = a_{12}a_{21} + \cdots - 3$$

$$\geq 2+1+1+2-3 = 3$$

but $t_1t_2t_3 \neq$ id $\Rightarrow \nexists$ finite order

$\Rightarrow W(w)$ is infinite.

$t_1t_2t_3t_4 = (\cdots)$ trace $= 6 > 4$

$\nexists$ finite order.

Lemma 2

Let $C$ be a symmetrizable cartan matrix of finite type, with corresponding Weyl group $W$.

Let $\Delta_C$ denote the root system corresponding to $C$,

$\Pi$ — simple roots

Then $\{m\alpha \mid m \in \mathbb{Z}, \alpha \in \Delta_C\}$ $C$ : [illegible]

$= \underline{\{\alpha \in \mathbb{Z}\Pi \mid W\alpha \subset \mathbb{N}_0\Pi \cup -\mathbb{N}_0\Pi\}}$

Proof of (ii)

Define a $\mathbb{Z}$-linear map $\phi : \mathbb{Z}\Pi \to \underline{\mathbb{Z}^n}$

where $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ — simple roots

$$\phi(\alpha_i) := e_i$$

Note that $\phi$ commutes with $s_i$

where $s_i$ are also interpreted as reflections on the set $\boxed{\delta_c}$

Since all element of $\delta_c$ - - - -

$$\Delta(B(V)) \underset{\boxed{=}}{\supset} \phi(\delta_c)$$

Since the multiplicities of the degree of generators of $B(V)$ are one, the multiplicity of $\phi(\alpha)$ is one.

Assume $\alpha \in \Delta(B(V)) \setminus \phi(\delta_c)$

It's well-known $\boxed{\alpha \neq 2e_i}$ $\forall i$.

height $= \infty$ but $\underline{\chi(e_i, e_i)}$ is $m$th primitive root.

$\underline{x_i^n = 0}$ ~~m≥~~ $n \geq m$

By {application of elements in $W$
one obtains

$$\alpha \notin \{m\phi(\beta) \mid m \in \mathbb{Z}, \beta \in \delta_c\}$$

~~if $\alpha \in \{m\phi(\beta) \mid m \in \mathbb{Z}, \beta \in \delta(\alpha)\}$,~~

by lemma 2, $\exists\, w \in W$ s.t

$$w\alpha \notin \underline{N_J^{\phi}(E) \cup -N_J^{\phi}(E)} - E_0$$

$$\underline{\{m\phi(\alpha) \mid m \in \mathbb{Z}, \alpha \in \delta_c\}}$$

$$\underline{= \{\beta \in \mathbb{Z}E_0 \mid W(\beta) \subset N_J E_0 \cup -N_J E_0\}}$$

$\exists\, w \in W.$ $w\alpha \notin$

$\swarrow$ to $\Delta^+(B(U)) \cup -\Delta^+(B(V))$

$= \Delta(B(V)) = w(\Delta(B(V)))$

By Prop1, $s_i \Delta(B(V)) \boxed{=} \underline{\Delta(B_i)} = \Delta(B(V))$

if $V$, $V'$ satisfy $\underline{q_{ij} q_{ji} = \hat{q}_{ij} \hat{q}_{ji}}$
$(q_{ij})$ $(\hat{q}_{ij})$

then we call twist equivalent,

$B(V) \cong B(V')$ as graded v.s

then $\Delta(B(V)) = \Delta(B(V'))$

Ex

Recall that $q_{ii} = q_{jj}$, $i, j$ are in the same connected component.

and any root $\alpha \in \Delta_c$ can be written as $\alpha = w(\alpha_i)$ for simple root $\alpha_i$, $w \in W$.

$\chi(\ ,\ )$ (2.4)